Extreme values of sectional curvature on the homogeneous complex manifolds $U(n+1)/U(n) \times U(p+1)/U(p)$

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Abstract

Invariant complex structures on the homogeneous manifold $U(n+1)/U(n)\times U(p+1)/U(p)$ are reseached. Extreme values of sectional curvature of Hermitian metrics on this manifold are found.

1. Hermitean structures on the $S^{2n+1} \times S^{2p+1}$

Recall some known construction of complex structures on $S^{2n+1} \times S^{2p+1}$ [2]. It is known that $S^{2n+1} \times S^{2p+1}$ is a principal $S^1 \times S^1$ bundle over $\mathbb{CP}^n \times \mathbb{CP}^p$. The space $\mathbb{CP}^n \times \mathbb{CP}^p$ and fiber $S^1 \times S^1$ are complex manifolds. If we fix complex structures on the base and fiber, then we can choose holomorphic transition functions to get complex structure on $S^{2n+1} \times S^{2p+1}$. All those structures form two parametric family $I(a,c)$ $(c>0)$.

The product $S^{2n+1} \times S^{2p+1}$ is homogeneous space $U(n+1)/U(n)\times U(p+1)/U(p)$. Let $\mathfrak{g}_1$ and $\mathfrak{h}_1$ ($\mathfrak{g}_2$ and $\mathfrak{h}_2$) are Lee algebras $\mathfrak{u}(n+1)$ and $\mathfrak{u}(n)$ ($\mathfrak{u}(p+1)$ and $\mathfrak{u}(p)$) of Lee groups $U(n+1)$ and $U(n)$ ($U(p+1)$ and $U(p)$). Let $\mathfrak{h} = \mathfrak{h}_1 \times \mathfrak{h}_2$. The complex structures $I(a,c)$ are $ad\mathfrak{h}$ - invariant [2].

Let $E^1_{\nu\mu}$ is matrix with 1 on the (ν,μ) - place and other zero elements. Define:

$$Z^1_{\nu\mu} = E^1_{\nu\mu} - E^1_{\mu\nu},\ T^1_{\nu\mu} = E^1_{\nu\mu} + E^1_{\mu\nu},\ 0 \le \mu < \nu \le n,$$

Take decomposition $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{p}_1$, where $\mathfrak{p}_1$ has basis $X^1 = \frac{1}{2}iT^1_{00}$, $Y^1_{2\nu-1} = Z^1_{\nu 0}$, $Y^1_{2\nu} = iT^1_{\nu 0}$. Remark that X^1 is tangent to fiber S^1 in the fibre bundle $S^{2n+1} \xrightarrow{S^1} \mathbb{CP}^n$. By analogy, we have $\mathfrak{g}_2 = \mathfrak{h}_2 \oplus \mathfrak{p}_2$ and $\mathfrak{g}_1 \times \mathfrak{g}_2 = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{p}_1 \times \mathfrak{p}_2$. So $S^{2n+1} \times S^{2p+1}$ viewed as homogeneous space $U(n+1)/U(n)\times U(p+1)/U(p)$ has basis $X^1, Y^1_{2\nu-1}, Y^1_{2\nu}, X^2, Y^2_{2\mu-1}, Y^2_{2\mu}$, $1 \le \nu \le n$, $1 \le \mu \le p$. On these basis vectors

$$I(a,c)X^1 = \frac{a}{c}X^1 + \frac{1}{c}X^2, \quad I(a,c)X^2 = -\frac{a^2+c^2}{c}X^1 - \frac{a}{c}X^2$$

$$I(a,c)Y^1_{2\nu-1} = Y^1_{2\nu}, \quad I(a,c)Y^2_{2\mu-1} = Y^2_{2\mu},$$

parameters a and c are real and $c > 0$.

Definition 1 *Almost complex structure J on the manifold M is called positive associated with 2-form ω if:*
1) $\omega(JX, JY) = \omega(X, Y)$,, for all $X, Y \in TM$
2) $\omega(X, JX) > 0$, for all nonzero $X \in TM$

Fix on the $S^{2n+1} \times S^{2p+1}$ nondegenerate invariant 2-form ω by:

$$\omega = X^1 \wedge X^2 + \sum_{\nu=1}^{n} Y^1_{2\nu-1} \wedge Y^1_{2\nu} + \sum_{\nu=1}^{p} Y^2_{2\nu-1} \wedge Y^2_{2\nu}$$

Lemma 1 *All complex structures $I(a,c)$ are positive associated with ω.*

Proof. For $I(a,c)$ properties 1) and 2) of definition 1 are obvious

Corollary 1 *Each complex structure $I(a,c)$ defines unique ω - associated metric by formula*

$$g(a,c)(X,Y) = \omega(X, I(a,c)Y)$$

These associated metrics in the above basis are:

$$g(a,c) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where

$$g_{11} = \begin{pmatrix} 1/c & 0_n^\top \\ 0_n & E_n \end{pmatrix}, \; g_{22} = \begin{pmatrix} (a^2+c^2)/c & 0_p^\top \\ 0_p & E_p \end{pmatrix},$$

$$g_{21}^\top = g_{12} = \begin{pmatrix} -a/c & 0_p^\top \\ 0_n & 0 \end{pmatrix}$$

0_n is zero column-vector, E_n is unit $n \times n$-matrix.

Each metric of this family is $I(a,c)$-Hermitean, so we obtain two-parametric family of Hermitean manifolds $(S^{2n+1} \times S^{2p+1},\ g(a,c), I(a,c), \omega)$. Invariant metric induces scalar product on $\mathfrak{p}$. We will denote this product as $\langle , \rangle_{a,c}$.

Proposition 1 *Invariant Riemannian connection for $g(a,c)$ on $S^{2n+1} \times S^{2p+1}$ is given by formula $D_X Y = \frac{1}{2}[X,Y]_{\mathfrak{p}} + U(X,Y)$, where U is symmetric bilinear mapping $U : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathfrak{p}$:*

$$U(X^1, Y^1_{2\nu-1}) = \frac{2-c}{2c}\, Y^1_{2\nu}, \; U(X^1, Y^1_{2\nu}) = -\frac{2-c}{2c}\, Y^1_{2\nu-1},$$

$$U(X^1,Y^2_{2\nu-1}) = -\frac{a}{c}\ Y^2_{2\nu},\ U(X^1,Y^2_{2\nu}) = \frac{a}{c}\ Y^2_{2\nu-1},$$
$$U(X^2,Y^1_{2\nu-1}) = -\frac{a}{c}\ Y^1_{2\nu},\ U(X^2,Y^1_{2\nu}) = \frac{a}{c}\ Y^1_{2\nu-1},$$
$$U(X^2,Y^2_{2\nu-1}) = \left(\frac{a^2+c^2}{c} - \frac{1}{2}\right)\ Y^2_{2\nu},\ U(X^2,Y^2_{2\nu}) = -\left(\frac{a^2+c^2}{c} - \frac{1}{2}\right)\ Y^2_{2\nu-1},$$

For other basis vectors X and Y the $U(X,Y)$ is equal to 0.

Proof. One can find U by formula: $2g(U(X,Y),Z) = g([Z,X]_{\mathfrak{p}},Y) + g(X,[Z,Y]_{\mathfrak{p}})$

Proposition 2 *Two-parametric family of metrics $g(a,c)$ has following characteristics:*
1) Ricci curvature in the above basis is:

$$Ric(a,c) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, r_{11} = \begin{pmatrix} 2\frac{n+pa^2}{c^2} & 0_n^\top \\ 0_n & 2(1+n-\frac{1}{c})E_n \end{pmatrix},$$

$$r_{22} = \begin{pmatrix} 2\frac{na^2+p(a^2+c^2)^2}{c^2} & 0_p^\top \\ 0_p & 2(1+p-\frac{a^2+c^2}{c})E_p \end{pmatrix},$$
$$r_{21}^\top = r_{12} = \begin{pmatrix} -2\frac{a}{c^2}(n+p(a^2+c^2)) & 0_p^\top \\ 0_n & 0 \end{pmatrix}.$$

Proper values of Ricci curvature $\tilde{r}_i$ are $\tilde{r}_{1,2} = \frac{x+y\pm\sqrt{(x-y^2+4z^2)}}{2}$, where $x = 2\frac{n+pa^2}{c^2}$, $y = 2\frac{na^2+p(a^2+c^2)^2}{c^2}$, $z = -2\frac{a}{c^2}(n+p(a^2+c^2))$; $\tilde{r}_3 = \tilde{r}_4 = \cdots = \tilde{r}_{2n+2} = 2(1+n-\frac{1}{c})$, $\tilde{r}_{2n+3} = \tilde{r}_{2n+4} = \cdots = \tilde{r}_{2n+2p+2} = 2(1+p-\frac{a^2+c^2}{c})$.
2). Scalar curvature is given by following formula:

$$s = 4n\left(1+n-\frac{1}{2c}\right) + 4p\left(1+p-\frac{a^2+c^2}{2c}\right)$$

Proof. Compute Ricci curvature by formula [1]

$$Ric(a,c)(X,X) = -\frac{1}{2}\sum_i |[X,v_i]_{\mathfrak{p}}|^2 - \frac{1}{2}\sum_i \langle [X,[X,v_i]_{\mathfrak{p}}]_{\mathfrak{p}}, v_i\rangle_{a,c}$$
$$-\sum_i \langle [X,[X,v_i]_{\mathfrak{h}}]_{\mathfrak{p}}, v_i\rangle_{a,c} + \frac{1}{4}\sum_{i,j}\langle [v_i,v_j]_{\mathfrak{p}}, X\rangle^2_{a,c} - \langle [Z,X]_{\mathfrak{p}}, X\rangle_{a,c}$$

where $Z = \sum_i U(v_i,v_i)$ and v_i is orthonormal basis of $(\mathfrak{p}, \langle,\rangle_{a,c})$. Scalar curvature is computed as a trace of Ricci curvature: $s = Ric_{ij}g^{ij}$, where g^{ij} are components of $g(a,c)^{-1}$ $(i,j = 1,\ldots,2n+2p+2)$.

2. Extreme values of sectional curvature of metrics $g(a,c)$.

Use bivectors to compute sectional curvature. Let $X_0,\dots,X_m$ is orthonormal basis of tangent bundle of any manifold. Let $B_{\nu\mu}=X_\nu\wedge X_\mu$ is basis of space of bivectors.

Let define:

$$R_{(\alpha\nu)(\rho\mu)}=\langle R(X_\alpha,X_\nu)X_\mu,X_\rho\rangle_{a,c}$$

R will denote $\frac{m(m+1)}{2}\times\frac{m(m+1)}{2}$ - matrix ($\frac{m(m+1)}{2}$ is dimension of space of bivectors) with element $R_{(\alpha\nu)(\rho\mu)}$ on the place $((\alpha\nu)(\rho\mu))$, where $\alpha<\nu$, and $\rho<\mu$.

As curvature is polylinear we have: sectional curvature in direction of unit bivector B is equal to

$$K(B)=\langle R\cdot B^T,B\rangle_{a,c},$$

where B^T is column-vector, which consists of bivector B coordinates, $R\cdot B^T$ is matrix-product, $\langle,\rangle_{a,c}$ is scalar product in the space of bivectors, corresponding to metric $\langle,\rangle_{a,c}$ on vectors.

Bivector B is called decomposable, if there exist vectors X and Y such as $B=X\wedge Y$. Note, that only decomposable bivectors correspond to 2-dimensional plane in the tangent bundle.

Compute matrix of curvature operator R of space $(S^{2n+1}\times S^{2p+1},g(a,c))$. Orthonormar basis of space $(\mathfrak{p},\langle,\rangle_{a,c})$ is

$$Z_0=\sqrt{c}X^1,\ Z_\nu=Y^1_\nu,\ \nu=1,\dots,2n$$

$$Z_{2n+1}=\frac{a}{\sqrt{c}}X^1+\frac{1}{\sqrt{c}}X^2,\ Z_{2n+1+\mu}=Y^2_\mu,\ \mu=1,\dots,2p$$

Proposition 3

$$R_{(0\nu)(\rho\mu)}=\begin{cases}\frac{1}{c},\ \rho=0,\ \nu=\mu=1,\dots,2n,\\ \frac{a^2}{c},\ \rho=0,\ \nu=\mu=2n+2,\dots,2n+2p+1,\\ -a,\ \rho=2n+1,\ \nu=\mu=2n+2,\dots,2n+2p+1,\\ 0,\ \textit{for other}\ \nu,\rho,\mu.\end{cases}$$

$$R_{(2n+1\nu)(\rho\mu)}=\begin{cases}-a,\ \rho=0,\ \nu=\mu=2n+2,\dots,2n+2p+1,\\ c,\ \rho=2n+1,\ \nu=\mu=2n+2,\dots,2n+2p+1,\\ 0,\ \textit{for other}\ \nu,\rho,\mu.\end{cases}$$

Proof. Compute $R_{(0\nu)(\rho\mu)}$. By definition

$$R_{(0\nu)(\rho\mu)}=-\sqrt{c}\langle R(X^1,Z_\nu)Z_\rho,Z_\mu\rangle_{a,c}.$$

$$R(X^1,Z_\nu)Z_\rho=D_{X^1}\left(\frac{1}{2}[Z_\nu,Z_\rho]_{\mathfrak{p}}+U(Z_\nu,Z_\rho)\right)-D_{Z_\nu}\left(\frac{1}{2}[X^1,Z_\rho]_{\mathfrak{p}}+U(X^1,Z_\rho)\right)$$

$$-\left(\frac{1}{2}[[X^1, Z_\nu]_\mathfrak{p}, Z_\rho]_\mathfrak{p} + U([X^1, Z_\nu]_\mathfrak{p}, Z_\rho)\right)$$

1) Let $\nu = 1, \dots, 2n$, study the case of $\nu = 2k$, nonzero values of R are possible in two cases only,when $\rho = 2k$ or $\rho = 2n+1$.
– $\rho = 2k$:

$$R(X^1, Y^1_{2k})Y^1_{2k} = -D_{Y^1_{2k}}\left(1 - \frac{1}{c}\right)Y^1_{2k-1} + X^1 = \frac{1}{c}X_1$$

– $\rho = 2n+1$:

$$R(X^1, Y^1_{2k})Z_{2n+1} = -D_{X^1}\left(-\frac{a}{2\sqrt{c}} + \frac{a(c-2)}{2c\sqrt{c}} + \frac{a}{c\sqrt{c}}\right)Y^1_{2k-1}$$

$$-\left(\frac{a}{2\sqrt{c}} + \frac{a(2-c)}{2c\sqrt{c}} - \frac{a}{c\sqrt{c}}\right)Y^1_{2k} = 0$$

By analogy, for $\nu = 2k-1$,

$$R(X^1, Y^1_{2k-1})Z_\rho = \begin{cases} \frac{1}{c}X^1, & \rho = 2k-1, \\ 0, & \rho \neq 2k-1. \end{cases}$$

So, for $\nu = 1, \dots, 2n$, $\rho \neq 1$:

$$R_{(0\nu)(\rho\mu)} = \begin{cases} \frac{1}{c}, & \rho = 0, \ \nu = \mu, \\ 0, & \text{in the otner cases.} \end{cases}$$

2) Let $\nu = 2n+1$, we have:

$$R(X^1, Z_{2n+1})Z_\rho = \frac{a}{\sqrt{c}}R(X^1, X^1)Z_\rho + \frac{1}{\sqrt{c}}R(X^1, X^2)Z_\rho = 0$$

3) Let $\nu = 2n+1+i$, where $i = 1, \dots, 2p$. Study the case $i = 2k-1$, we have:

$$R(X^1, Y^2_{2k-1})Z_\rho = D_{X^1}(\frac{1}{2}[Y^2_{2k-1}, Z_\rho]_\mathfrak{p} + U(Y^2_{2k-1}, Z_\rho)) - D_{Y^2_{2k-1}}(\frac{1}{2}[X^1, Z_\rho]_\mathfrak{p} + U(X^1, Z_\rho))$$

Nonzero values of R are possible in cases $\rho = 2n+1$ or $\rho = 2n+2k$:
– $\rho = 2n+1$:

$$R(X^1, Y^2_{2k-1})Z_{2n+1} = D_{X^1}(\frac{1}{2}[Y^2_{2k-1}, \frac{1}{\sqrt{c}}X^2]_\mathfrak{p} + U(Y^2_{2k-1}, \frac{a}{\sqrt{c}}X^1 + \frac{1}{\sqrt{c}}X^2)) = \frac{a}{\sqrt{c}}Y^2_{2k-1}$$

– $\rho = 2n+2k$:

$$R(X^1, Y^2_{2k-1})Z_{2n+2k} = -D_{Y^2_{2k-1}}U(X^1, Y^2_{2k-1}) = -\frac{a}{c}X^2$$

By analogy, for $i = 2k$ we have:

$$R(X^1, Y^2_{2k})Z_{2n+1} = \frac{a}{\sqrt{c}}Y^2_{2k}, \qquad R(X^1, Y^2_{2k})Z_{2n+2k+1} = -\frac{a}{c}Y^2_{2k}$$

So, for $\nu = 2n+2, \dots, 2n+2p+1$

$$R_{(0\nu)(\rho\mu)} = \begin{cases} \frac{a^2}{c}, & \rho = 0,\ \nu = \mu, \\ -a, & \rho = 2n+1,\ \mu = \nu, \\ 0, & \rho \neq 2n+1,\ \rho \neq 0. \end{cases}$$

Compute $R_{(2n+1\nu)(\rho\mu)}$. By definition:

$$R_{(2n+1\nu)(\rho\mu)} = -\langle R(Z_{2n+1}Z_\nu)Z_\rho, Z_\mu\rangle_{a,c} = -\frac{a}{\sqrt{c}}\langle R(X^1, Z_\nu)Z_\rho, Z_\mu\rangle_{a,c} - \frac{1}{\sqrt{c}}\langle R(X^2, Z_\nu)Z_\rho, Z_\mu\rangle_{a,c}$$

Let $\nu = 2n+1+i$, as:

$$R_{(2n+1\nu)(\rho\mu)} = R_{(\rho\mu)(2n+1\nu)}$$

it is enough to research case $\rho > 0$.
1) $\rho = 1, \dots, 2n$

$$R(X^1, Y^2_i)Y^1_\rho = -D_{Y^2_i}\left(\frac{1}{2}[X^1, Y^1_\rho]_{\mathfrak{p}} + U(X^1, Y^1_\rho)\right) = 0$$

$$R(X^2, Y^2_i)Y^1_\rho = -D_{Y^2_i}U(X^2, Y^1_\rho) = 0$$

2) $\rho = 2n+1$
Values of $R(X^1, Y^2_i)Z_{2n+1}$ are obtained above.

$$R(X^2, Y^2_i)Z_{2n+1} = \frac{a}{\sqrt{c}}R(X^2, Y^2_i)X^1 + \frac{1}{\sqrt{c}}R(X^2, Y^2_i)X^2$$

$$R(X^2, Y^2_i)X^1 = D_{X^2}U(Y^2_i, X^1) - U([X^2, Y^2_i]_{\mathfrak{p}}, X^1) =$$

$$= \begin{cases} i = 2k-1: & -\frac{a}{c}D_{X^2}Y^2_{2k} + U(Y^2_{2k}, X^1) = \frac{a(a^2+c^2)}{c^2}Y^2_{2k-1}, \\ i = 2k: & \frac{a}{c}D_{X^2}Y^2_{2k-1} - U(Y^2_{2k-1}, X^1) = \frac{a(a^2+c^2)}{c^2}Y^2_{2k}. \end{cases}$$

$$R(X^2, Y^2_i)X^2 = D_{X^2}\left(\frac{1}{2}[Y^2_i, X^2]_{\mathfrak{p}} + U(Y^2_i, X^2)\right) - \frac{1}{2}[[X^2, Y^2_i]_{\mathfrak{p}}, X^2]_{\mathfrak{p}} - U([X^2, Y^2_i]_{\mathfrak{p}}, X^2)$$

$$= \begin{cases} i = 2k-1: & \frac{a^2+c^2}{c}D_{X^2}Y^2_{2k} - \frac{a^2+c^2}{c}Y^2_{2k-1} = -\frac{(a^2+c^2)^2}{c^2}Y^2_{2k-1}, \\ i = 2k: & -\frac{a^2+c^2}{c}D_{X^2}Y^2_{2k-1} - \frac{a^2+c^2}{c}Y^2_{2k} = -\frac{(a^2+c^2)^2}{c^2}Y^2_{2k}. \end{cases}$$

We obtain:

$$R(X^2, Y^2_i)Z_{2n+1} = -\frac{a^2+c^2}{\sqrt{c}}Y^2_i$$

We have:

$$R_{(2n+1\nu)(2n+1\mu)} = -\frac{a}{\sqrt{c}}\left(\frac{a}{\sqrt{c}}Y_i^2, Z_\mu\right) - \frac{1}{\sqrt{c}}\left(-\frac{a^2+c^2}{\sqrt{c}}Y_i^2, Z_\mu\right)$$
$$= \begin{cases} c, & \mu = 2n+1+i, \\ 0, & \mu \neq 2n+1+i. \end{cases}$$

3) Let $\rho = 2n+1+j$

$$R(X^1, Y_i^2)Y_j^2 = -D_{Y_i^2}U(X^1, Y_j^2) = \begin{cases} -\frac{a}{c}X^2, & i=j, \\ 0, & i \neq j. \end{cases}$$

$$R(X^2, Y_i^2)Y_j^2 = -D_{Y_i^2}\left(\frac{1}{2}[X^2, Y_j^2]_{\mathfrak{p}} + U(X^2, Y_j^2)\right) - \frac{1}{2}[[X^2, Y_i^2]_{\mathfrak{p}}, Y_j^2]_{\mathfrak{p}} - U([X^2, Y_i^2]_{\mathfrak{p}}, Y_j^2)$$
$$= \begin{cases} \frac{a^2+c^2}{c}X^2, & i=j, \\ 0, & i \neq j. \end{cases}$$

So, for $\rho = 2n+2, \dots, 2n+2p+1$ $R_{(2n+1\nu)(\rho\mu)} = 0$, if $\mu > \rho$.

Let B is some decomposable bivector, having coordinates $(b_{01}, \dots, b_{2n+2p2n+2p+1})$ in the basis $Z_\nu \wedge Z_\mu$. Decompose B in the following sum:

$$B = B' + B''$$

where
$B' = (b_{01}, \dots, b_{02n}, 0, b_{02n+2}, \dots, b_{02n+2p+1}, 0, \dots, 0, b_{2n+12n+2}, \dots, b_{2n+12n+2p+1}, 0, \dots, 0)$,
$B'' = B - B'$. By proposition 3:

$$K(B) = \langle R \cdot B^T, B\rangle_{a,c} = \langle R \cdot B'^T, B'\rangle_{a,c} + \langle R \cdot B''^T, B''\rangle_{a,c}$$

If decompose B' in the sum:

$$B' = B'_1 + B'_2 + B'_3,$$

where
$B'_1 = (b_{01}, \dots, b_{02n}, 0, \dots, 0)$,
$B'_2 = (0, \dots, 0, b_{02n+2}, \dots, b_{02n+2p+1}, 0, \dots, 0)$,
$B'_3 = (0, \dots, 0, b_{2n+12n+2}, \dots, b_{2n+12n+2p+1}, 0, \dots, 0)$, then:

$$K(B) = \frac{1}{c}\|B'_1\|^2 + \frac{a^2}{c}\|B'_2\|^2 + c\|B'_3\|^2 - 2a\sum_{i=1}^{p} b_{02n+1+i}b_{2n+12n+1+i} + \langle R \cdot B''^T, B''\rangle_{a,c}$$

Proposition 4 $R_{(2l-12l)(2n+2m2n+2m+1)} = \frac{2a}{c}$
$R_{(\alpha\nu)(\rho\mu)} = 0$ *for other* $\alpha, \nu = 1, \dots, 2n,$ $\rho, \mu = 2n+2, \dots, 2n+2p+1$

Proof. Let $\alpha, \nu = 1, \dots, 2n$, $\rho = 2n+1+i$, $\mu = 2n+1+j$, $i, j = 1, \dots, 2p$, then:

$$R_{(\alpha\nu)(\rho\mu)} = \langle R(Y^1_\alpha, Y^1_\nu)Y^2_j, Y^2_i\rangle_{a,c}$$

$$R(Y^1_\alpha, Y^1_\nu)Y^2_j = -\frac{1}{2}[[Y^1_\alpha, Y^1_\nu]_{\mathfrak{p}}, Y^2_j]_{\mathfrak{p}} - U([Y^1_\alpha, Y^1_\nu]_{\mathfrak{p}}, Y^2_j) =$$
$$= \begin{cases} 2U(X^1, Y^2_j) & \alpha = 2l-1,\ \nu = 2l, \\ 0, & \text{for other } \alpha,\ \nu. \end{cases}$$

As

$$2U(X^1, Y^2_j) = \begin{cases} \frac{2a}{c}Y^2_{2m-1}, & j = 2m, \\ -\frac{2a}{c}Y^2_{2m}, & j = 2m-1. \end{cases},$$

then

$$R_{(2l-12l)(2n+2m2n+2m+1)} = \frac{2a}{c}$$

Proposition 5

$$R_{(2l-12n+2m)(2l2n+2m+1)} = \frac{a}{c}$$
$$R_{(2l2n+2m)(2l-12n+2m+1)} = -\frac{a}{c}$$
$$R_{(2l-12n+2m+1)(2l2n+2m)} = -\frac{a}{c}$$
$$R_{(2l2n+2m+1)(2l-12n+2m)} = \frac{a}{c}$$
$$R_{(\alpha\nu)(\rho\mu)} = 0$$

for other values of $\alpha, \rho = 1, \dots, 2n$, $\nu, \mu = 2n+2, \dots, 2n+2p+1$

Proof. Let $\alpha, \rho = 1, \dots, 2n$, $\nu = 2n+1+i$, $\mu = 2n+1+j$.

$$R_{(\alpha\nu)(\rho\mu)} = \langle R(Y^1_\alpha, Y^2_i)Y^2_j, Y^1_\rho\rangle_{a,c}$$

$$R(Y^1_\alpha, Y^2_i)Y^2_j = \frac{1}{2}D_{Y^1_\alpha}[Y^2_i, Y^2_j]_{\mathfrak{p}} = \begin{cases} -D_{Y^1_\alpha}X^2, & i = 2m-1,\ j = 2m, \\ D_{Y^1_\alpha}X^2, & i = 2m,\ j = 2m-1, \\ 0, & |j-i| \neq 1 \end{cases}$$
$$= \begin{cases} -U(Y^1_{2l-1}, X^2) = \frac{a}{c}Y^1_{2l}, & \alpha = 2l-1,\ i = 2m-1,\ j = 2m, \\ -U(Y^1_{2l}, X^2) = -\frac{a}{c}Y^1_{2l}, & \alpha = 2l,\ i = 2m-1,\ j = 2m, \\ U(Y^1_{2l-1}, X^2) = -\frac{a}{c}Y^1_{2l}, & \alpha = 2l-1,\ i = 2m,\ j = 2m-1, \\ U(Y^1_{2l}, X^2) = \frac{a}{c}Y^1_{2l}, & \alpha = 2l,\ i = 2m,\ j = 2m-1, \\ 0, & |j-i| \neq 1. \end{cases}$$

Decompose B'' in sum of three bivectors:

$$B'' = B''_1 + B''_2 + B''_3$$

where B''_1 is bivector with nonzero coordinates b_{ij}, $i = 1, \ldots, 2n$, $j = 2n+2, \ldots, 2n+2p+1$,
B''_2 is bivector with nonzero coordinates b_{ij}, $i, j = 1, \ldots, 2n$,
B''_3 is bivector with nonzero coordinates b_{ij}, $i, j = 2n+2, \ldots, 2n+2p+1$.
Then by results of propositions 4 and 5 we have:

$$\langle R\cdot B''^T, B''\rangle_{a,c} = \langle R\cdot B''^T_1, B''_1\rangle_{a,c} + \langle R\cdot B''^T_2, B''_3\rangle_{a,c} + \langle R\cdot B''^T_3, B''_2\rangle_{a,c} + \langle R\cdot B''^T_2, B''_2\rangle_{a,c}$$

$$+\langle R\cdot B''^T_3, B''_3\rangle_{a,c} = 2\frac{a}{c}\sum_{l=1}^{n}\sum_{m=1}^{p}(b_{2l-12n+2m}b_{2l2n+2m+1} - b_{2l2n+2m}b_{2l-12n+2m+1})$$

$$+4\frac{a}{c}\sum_{l=1}^{n}\sum_{m=1}^{p} b_{2l-12l}b_{2n+2m2n+2m+1} + \langle R\cdot B''^T_2, B''_2\rangle_{a,c} + \langle R\cdot B''^T_3, B''_3\rangle_{a,c}$$

As $\langle R\cdot B''^T_2, B''_2\rangle_{a,c} = K(B''_2)$ and $\langle R\cdot B''^T_3, B''_3\rangle_{a,c} = K(B''_3)$ we can use the results of work of Volper D.E. [3] and obtain:

$$\min(4-\frac{3}{c};1)\|B''_2\|^2 \le K(B''_2) \le \max(4-\frac{3}{c};1)\|B''_2\|^2$$

$$\min(4-\frac{3(a^2+c^2)}{c};1)\|B''_3\|^2 \le K(B''_3) \le \max(4-\frac{3}{c};1)\|B''_3\|^2$$

Theorem 1 *Sectional curvature $K(a,c)$ of metric $g(a,c)$ on $S^{2n+1}\times S^{2p+1}$ satisfies to the following inequality:*

$$K_{\min} \le K(a,c) \le K_{\max}$$

where

$$K_{\min} = \begin{cases} \min(-|\frac{a}{c}|, \frac{5c-3-\sqrt{16a^2-18c+9c^2+9}}{2c}) & a^2+(c-\frac{1}{2})^2 \le \frac{1}{4} \\ \min(-|\frac{a}{c}|, \frac{8c-3(1+a^2+c^2)-\sqrt{9(a^2+c^2-1)^2+16a^2}}{2c}) & a^2+(c-\frac{1}{2})^2 > \frac{1}{4}, c<1 \\ \min(-|\frac{a}{c}|, \frac{5c-3(a^2+c^2)-\sqrt{16a^2+9c^2-18c(a^2+c^2)+9(a^2+c^2)^2}}{2c}) & c \ge 1 \end{cases}$$

$$K_{\max} = \begin{cases} \max(\frac{1}{c}, \frac{5c-3(a^2+c^2)+\sqrt{16a^2+9c^2-18c(a^2+c^2)+9(a^2+c^2)^2}}{2c}) & a^2+(c-\frac{1}{2})^2 \le \frac{1}{4} \\ \max(\frac{a^2+c^2}{c}, 1+2|\frac{a}{c}|, \frac{1}{c}) & a^2+(c-\frac{1}{2})^2 > \frac{1}{4}, c<1 \\ \max(\frac{a^2+c^2}{c}, \frac{5c-3+\sqrt{16a^2-18c+9c^2+9}}{2c}) & c \ge 1 \end{cases}$$

Proof. As above results we have

$$K' \le K(a,c) \le K''$$

where

$$K' = \min(4-\frac{3}{c},1)\|B_2''\|^2+\min(4-3\frac{a^2+c^2}{c},1)\|B_3''\|^2+\frac{1}{c}\|B_1'\|^2+\frac{a^2}{c}\|B_2'\|^2+c\|B_3'\|^2$$

$$-2a\sum_{m=1}^{p} b_{2n+12n+1+m}b_{02n+1+m} + 4\frac{a}{c}\sum_{l=1}^{n}\sum_{m=1}^{p} b_{2l-12l}b_{2n+2m2n+2m+1}$$

$$+2\frac{a}{c}\sum_{l=1}^{n}\sum_{m=1}^{p}(b_{2l-12n+2m}b_{2l2n+2m+1} - b_{2l2n+2m}b_{2l-12n+2m+1})$$

$$K'' = \max(4-\frac{3}{c},1)\|B_2''\|^2+\max(4-3\frac{a^2+c^2}{c},1)\|B_3''\|^2+\frac{1}{c}\|B_1'\|^2+\frac{a^2}{c}\|B_2'\|^2+c\|B_3'\|^2$$

$$-2a\sum_{m=1}^{p} b_{2n+12n+1+m}b_{02n+1+m} + 4\frac{a}{c}\sum_{l=1}^{n}\sum_{m=1}^{p} b_{2l-12l}b_{2n+2m2n+2m+1}$$

$$+2\frac{a}{c}\sum_{l=1}^{n}\sum_{m=1}^{p}(b_{2l-12n+2m}b_{2l2n+2m+1} - b_{2l2n+2m}b_{2l-12n+2m+1})$$

It is obvious that

$$\min(4-\frac{3}{c},1) = \begin{cases} 4-\frac{3}{c}, & c<1, \\ 1, & c\geq 1. \end{cases}$$

$$\min(4-3\frac{a^2+c^2}{c},1) = \begin{cases} 1, & a^2+(c-\frac{1}{2})^2 < \frac{1}{4}, \\ 4-3\frac{a^2+c^2}{c}, & a^2+(c-\frac{1}{2})^2 \geq \frac{1}{4}. \end{cases}$$

Let, for example, $a^2+(c-\frac{1}{2})^2 \leq \frac{1}{4}$. Then $\min(4-\frac{3}{c},1) = 4-\frac{3}{c}$, $\min(4-3\frac{a^2+c^2}{c},1)=1$. As $\|B\|=1$, we have to solve problem of conditional extremum of function $K'(B)$. We obtain:

$$K_{\min} = \min_{\|B\|=1} K'(B) = \min(\pm\frac{a}{c},0,\frac{a^2+c^2}{c},\frac{1}{c},\frac{5c-3\pm\sqrt{16a^2-18c+9c^2+9}}{2c},4-3\frac{1}{c},1)$$

$$= \min(-|\frac{a}{c}|,\frac{5c-3-\sqrt{16a^2-18c+9c^2+9}}{2c})$$

By analogy

$$K_{\max} = \max_{\|B\|=1} K''(B)$$

$$= \max(\pm\frac{a}{c},0,\frac{a^2+c^2}{c},\frac{1}{c},\frac{5c-3(a^2+c^2)\pm\sqrt{16a^2+9c^2-18c(a^2+c^2)+9(a^2+c^2)^2}}{2c},$$

$$4-3\frac{a^2+c^2}{c},1) = \max(\frac{1}{c},\frac{5c-3(a^2+c^2)+\sqrt{16a^2+9c^2-18c(a^2+c^2)+9(a^2+c^2)^2}}{2c})$$

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